

Harmonic-Counting Measures and Spectral Theory of Lens Spaces

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Abstract

In this article, associated with each lattice $T \subseteq \mathbb{Z}^n$ the concept of a harmonic-counting measure ν_T on a sphere S^{n-1} is introduced and it is applied to determine the asymptotic behavior of the eigenfunctions of the Laplace-Beltrami operator on a lens space. In fact, the asymptotic behavior of the cardinality of the set of independent eigenfunctions associated with the elements of T which lie in a cone is determined when T is the lattice of a lens space.

Keywords: Counting measure, Lens space, Lattice, Isospectrality, Laplace-Beltrami operator.

1 Introduction

Counting the number of points of a lattice in a convex body is well studied by many mathematicians including Minkowski, Ehrhart and Stanley. The asymptotic behavior of such counting functions leads to the definition of lattice-counting-measures on the sphere S^{n-1} [2, 3, 9]. In this paper we define the parallel notion of a harmonic counting measure. Let H be the vector space of $2n$ -variable harmonic polynomials that are invariant under the action $(z_1, z_2, \dots, z_n) \rightarrow (e^{2\pi \frac{p_1}{q}} z_1, \dots, e^{2\pi \frac{p_n}{q}} z_n)$ where q and the p_i belong to \mathbb{N} and we identify \mathbb{R}^{2n} with \mathbb{C}^n . H can be decomposed in a natural way into the vector spaces $H_{s,(a_1,\dots,a_n)}$, which consist of harmonic homogeneous polynomials of degree s associated with $(a_1, \dots, a_n) \in T = \{(a_1, \dots, a_n) \in (\mathbb{N} \cup \{0\})^n \mid \sum_{j=1}^n a_j p_j \equiv 0 \pmod{q}\}$ [8, 7]. Asymptotic behavior of $F_{T \cap K}(t) = \sum_{s=0}^t \sum_{x \in T \cap K} \dim H_{s,x}$ leads to the definition of a harmonic counting measure where K is an l_1 -spherical cone. The natural relation between lattices, harmonic polynomials and the topological lens spaces is introduced in [7, 5, 4]. In these articles it is shown that some conditions on lattices determine the isospectrality and isometry of lens spaces. Also, it is shown that harmonic homogeneous polynomials that are invariant under the action of homotopy group of a lens space determine eigenspaces of the Laplace-Beltrami operator. Succinctly, in the present article we define harmonic counting measures and determine their relation to lattice-counting-measures and isospectrality of lens spaces.

2 Preliminaries on lens spaces

2.1 Lattices

A lattice T is a subgroup of the group \mathbb{Z}^n . T is of rank n if $T \otimes \mathbb{R} = \mathbb{R}^n$.

Definition 1. A preliminary lattice group T is defined as

$$T = \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid \sum_{j=1}^n a_j p_j \equiv 0 \pmod{q}\}$$

where integers $\{p_1, \dots, p_n\}$ are prime to the positive integer q .

The measures defined in this article can be used in general lattices. But we limit ourselves to preliminary lattices that are useful for the study of lens spaces. Let $\{v_1, \dots, v_n\}$ be a basis for T . The matrix A whose columns are v_1, \dots, v_n is called a generating matrix of T . B is another generating matrices of T iff there is a unimodular matrix U such that $A = UB$. An essential parallelepiped of a lattice $T \subset \mathbb{R}^n$ is a parallelepiped $P_T = \{\sum_{i=1}^n a_i v_i | 0 \leq a_i \leq 1, i = 1, \dots, n\}$. Let K be a cone in \mathbb{R}^n whose apex is the origin.

2.2 Harmonic counting measure

Let $N_{T \cap K}(s)$ be the number of elements in $T \cap K$ with the l_1 -norm s . For a cone $K \subset (\mathbb{R}^+)^n$ set

$$F_{T \cap K}(t) = \sum_{s=0}^t \sum_{r=0}^{\lfloor \frac{s}{2} \rfloor} \binom{r+n-2}{n-2} N_{T \cap K}(s-2r).$$

Definition 2. The cone constructed from a set $U \subseteq \mathbb{R}^n$ is the set $\{tx | t \in \mathbb{R}^+, x \in U\}$. This set is denoted by $C(U)$.

Definition 3. The harmonic counting measure associated with the lattice T , is a measure ν_T on the Borel σ -algebra of S^{n-1} which is defined as

$$\nu_T(U) := \lim_{t \rightarrow \infty} \frac{F_{T \cap C(U)}(t)}{t^{2n-1}}, \quad (1)$$

where $C(U) \subset (\mathbb{R}^+)^n$. And, it is extended symmetrically to the other Borel subsets.

By lemma 3.6 of [7],

$$\dim H_{s, (a_1, \dots, a_n)} = \begin{cases} \binom{r+n-2}{n-2} & \text{if } \|(a_1, \dots, a_n)\|_{l_1} = s-2r \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

and it is equal to the number of independent harmonic homogeneous polynomials of degree s associated with the element (a_1, \dots, a_n) . So the resulting measure is named harmonic-counting measure.

Theorem 1. ν_T is a finite measure and its total value is equal to

$$\frac{1}{q} 2^{1-n} \pi^{1-2n} \omega_{2n-1} \text{Vol}(S^{2n-1}).$$

This is a corollary of Theorem 4 but in the next section we provide another proof for preliminary lattices using the properties of lens spaces.

In order to study the asymptotic behavior of the function $F_{T \cap K}(t)$, we need the asymptotic behavior of $N_{\mathbb{Z}^n \cap K}(t)$. This is a well-known fact that $\sum_{t=0}^s N_{\mathbb{Z}^n \cap K}(t) \sim \alpha_K s^n$ where α_K is the volume of the intersection of K and the l_1 -sphere of radius 1 (Ehrhart-Stanley-Minkowski). This provides a combinatorial approach to a well-known measure μ_T on the sphere S^{n-1} [2]. Precisely

$$\mu_T(U) = \lim_{s \rightarrow \infty} \frac{\sum_{t=0}^s N_{\mathbb{Z}^n \cap A^{-1}C(U)}(t)}{s^n} \quad (3)$$

is a finite measure, when A is the generating matrix of T .

2.3 Lens spaces

Let q be a positive integer, and let p_1, \dots, p_n be integers that are prime to q . Let

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \sim e^{i\theta} \quad (4)$$

and

$$g = R(p_1/q) \oplus \dots \oplus R(p_n/q). \quad (5)$$

Suppose that $G \subset O(2n)$ is the finite cyclic group generated by g . If G (as a group of isometries) acts freely on S^{2n-1} , then the manifold S^{2n-1}/G , denoted by $\mathfrak{L}(p_1, \dots, p_n; q)$, is called a lens space. Let $\text{spec}(M)$ denote the set of eigenvalues of the Laplace-Beltrami operator. $G_1 \subseteq G$ implies $\text{spec}(S^{2n-1}/G) \subseteq \text{spec}(S^{2n-1}/G_1)$. In particular $\text{spec}(S^{2n-1}/G) \subseteq \text{spec}(S^{2n-1})$. The Laplace-Beltrami eigenvalues of the manifold S^{2n-1} are $k(k+2n-2)$, $k \in \mathbb{N} \cup \{0\}$ [5, 4].

Definition 4. The lens space associated with a lattice $T = \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid \sum_{j=1}^n a_j p_j \equiv 0 \pmod{q}\}$ is the space S^{2n-1}/G [7].

A nice relation between lattices and isospectrality is:

Theorem 2. (*Lauret, Miatello and Rossetti [7]*) Two lens spaces $\mathfrak{L}_1 = S^{2n-1}/G_1$ and $\mathfrak{L}_2 = S^{2n-1}/G_2$ are isospectral iff for the associated lattices T_1 and T_2 , $B_{l_1}(0, k) \cap T_1 = B_{l_1}(0, k) \cap T_2$ for each $k \in \mathbb{N}$ where $B_{l_1}(0, k)$, is the l_1 -ball of radius k .

For a relatively elementary proof see [8].

Theorem 3. Let \mathfrak{L}_1 and \mathfrak{L}_2 be two isospectral lens spaces with associated lattices T_1 and T_2 . Then $\mu_{T_1} = \mu_{T_2}$

Proof. It is well-known that for an arbitrary convex polytope $\Omega \subset \mathbb{R}^n$ we have $\lim_{s \rightarrow \infty} \frac{\text{card}(\mathbb{Z}^n \cap s\Omega)}{s^n} = \text{Vol}(\Omega)$ [1]. If A is a generating matrix of the lattice T and K is the part of $B_{l_1}(0, 1)$ opposite to $U \subseteq S^{n-1}$, then $\text{card}(T \cap sK) = \text{card}(\mathbb{Z}^n \cap sA^{-1}K)$. Therefore

$$\mu_T(U) = \lim_{s \rightarrow \infty} \frac{\text{card}(\mathbb{Z}^n \cap sA^{-1}K)}{s^n} = \text{Vol}(A^{-1}K) = \det A^{-1} \text{Vol}(K). \quad (6)$$

On the other hand by Theorem 2, the value of $\det A_1^{-1} \text{Vol}(B_{l_1}(0, 1))$ is equal to $\det A_2^{-1} \text{Vol}(B_{l_1}(0, 1))$. Therefore, the determinants of the generating matrices A_1 of T_1 and A_2 of T_2 are equal [8, 6]. Thus these measures are equivalent. \square

Proof of theorem 1:

Proof. Let T be a preliminary lattice and let \mathfrak{L} be its associated lens space. Also, let K^+ be the cone of elements of \mathbb{R}^n with positive coordinates. According to [7](or [8]) the number of independent eigenfunctions of the Laplace-Beltrami operator on a lens space with the eigenvalue $s(s+(2n-1)-1)$ is equal to

$$\sum_{r=0}^{\lfloor \frac{s}{2} \rfloor} \binom{r+n-2}{n-2} N_{T \cap K^+}(s-2r).$$

So $F_{T \cap K^+}(t)$ is equal to the number of independent eigenfunctions with eigenvalues less than $t(t+2n-2)$. By the Weyl law we have

$$\lim_{x \rightarrow \infty} \frac{N(x)}{x^{\frac{2d-1}{2}}} = (2\pi)^{-(2d-1)} \omega_{2d-1} \text{Vol}(M),$$

where $N(x)$ denotes the number of eigenvalues less than x and d is dimension of the manifold M . So

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{F_{T \cap K^+}(t)}{t^{2n-1}} &= \lim_{t \rightarrow \infty} \frac{N(t(t+2n-2))}{t^{2n-1}} = \lim_{t \rightarrow \infty} \frac{N(t(t+2n-2))}{(t(t+2n-2))^{\frac{2n-1}{2}}} \\ &= (2\pi)^{-(2n-1)} \omega_{2n-1} \text{Vol}(S^{2n-1}/G). \end{aligned} \quad (7)$$

S^{2n-1} is a q -sheeted covering space of S^{2n-1}/G and therefore $\text{Vol}(S^{2n-1}/G) = \frac{1}{q} \text{Vol}(S^{2n-1})$. We have 2^n parts for our coordinate, so the achieved number must be multiplied by 2^n . \square

Now we compute the value of $\nu_T(U)$ where U is a Borel subset of the sphere S^{n-1} . Let A be the generating matrix of T .

Theorem 4. *The value of $\nu_T(U)$ is equal to*

$$\lim_{t \rightarrow \infty} \frac{F_{T \cap C(U)}(t)}{t^{2n-1}} = \frac{B(n-1, n+1)}{(n-2)!2^{n-1}} \text{Vol}(A^{-1}(C(U)) \cap B_{l_1}(0, 1)), \quad (8)$$

where the beta function is defined as $B(z, t) = \int_0^1 x^{z-1}(1-x)^{t-1} dx$.

Proof. We have

$$F_{T \cap C(U)}(t) = \sum_{s=0}^t \sum_{r=0}^{\lfloor \frac{s}{2} \rfloor} \binom{r+n-2}{n-2} N_{T \cap C(U)}(s-2r), \quad (9)$$

where

$$\sum_{s=0}^t N_{T \cap C(U)}(s) = \text{Vol}(A^{-1}(C(U)) \cap B_{l_1}(0, 1)) t^n + O(t^{n-1}) = \alpha t^n + O(t^{n-1}). \quad (10)$$

By changing the order of summation in 9, we have

$$F_{T \cap C(U)}(t) = \sum_{r=0}^{\lfloor \frac{t}{2} \rfloor} \left(\binom{r+n-2}{n-2} \sum_{i=0}^{t-2r} N_{T \cap C(U)}(i) \right).$$

So by 10,

$$\begin{aligned} \frac{1}{t^{2n-1}} \sum_{r=0}^{\lfloor \frac{t}{2} \rfloor} \left(\binom{r+n-2}{n-2} \alpha (t-2r)^n - M (t-2r)^{n-1} \right) &\leq \\ \frac{1}{t^{2n-1}} \sum_{r=0}^{\lfloor \frac{t}{2} \rfloor} \left(\binom{r+n-2}{n-2} \sum_{i=0}^{t-2r} N_{T \cap C(U)}(i) \right) &\leq \quad (**) \\ \frac{1}{t^{2n-1}} \sum_{r=0}^{\lfloor \frac{t}{2} \rfloor} \left(\binom{r+n-2}{n-2} (\alpha (t-2r)^n + M (t-2r)^{n-1}) \right). \end{aligned}$$

Also we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=0}^t g\left(\frac{i}{t}\right) = \int_0^1 g(x) dx, \quad (11)$$

Applying (11) the limits of the left and the right parts of (**) are equal to $\frac{\alpha}{(n-2)!2^{n-1}} \int_0^1 x^{n-2}(1-x)^n dx$. So,

$$\lim_{t \rightarrow \infty} \frac{\sum_{r=0}^{\lfloor \frac{t}{2} \rfloor} \left(\binom{r+n-2}{n-2} \sum_{i=0}^{t-2r} N_{T \cap C(U)}(i) \right)}{t^{2n-1}} = \frac{\alpha}{(n-2)!2^{n-1}} B(n-1, n+1).$$

□

This shows that the normalization of ν_T is a uniform measure with respect to surface area on each face of $B_{l_1}(0, 1)$.

Remark 1. When \mathfrak{L} is the lens space associated with the lattice T , the set of independent eigenfunctions of the Laplace-Beltrami operator on \mathfrak{L} associated with the elements of $T \cap C(U) \cap B_{l_1}(0, m(m+2n-2))$ is the same as $F_{C(U) \cap T}(m)$. So, Theorem 4 determines the asymptotic behavior of the cardinality of eigenfunctions in each direction. Therefore, it provides more information than Weyl law for Laplace-Beltrami operator in the case of lens spaces.

Remark 2. Theorem 4 shows that the number of independent eigenfunctions of the Laplace-Beltrami operator associated with the integral points of $C(U) \cap B_{l_1}(0, t)$ is asymptotically $\frac{B(n-1, n+1)}{(n-2)!2^{n-1}} t^{n-1}$ times the number of lattice points in $C(U) \cap B_{l_1}(0, t)$.

Remark 3. Harmonic-counting measures are constant multiples of lattice-counting measures where the constant is an explicit function of dimension of the lattice.

Remark 4. A harmonic complex polynomial (invariant under the action of z_p) can be written uniquely as the summation of harmonic polynomials of the form $Q(|z_1|^2, \dots, |z_n|^2) z_1^{a_1} \dots z_n^{a_n}$ where $(a_1, \dots, a_n) \in T$ (the lattice associated to the action which is defined the same as the lattice associated to a lens space) and $Q(w_1, \dots, w_n)$ is a homogeneous polynomial [8]. Now let no multiple of (a_1, \dots, a_n) by an element $0 < t < 1$ belongs to the lattice T . The Theorem 4 asymptotically determines the number of independent homogeneous polynomials Q whose multiplication (after replacing (w_1, \dots, w_n) by $(|z_1|^2, \dots, |z_n|^2)$) by an integer power of $z_1^{a_1} \dots z_n^{a_n}$ is a harmonic polynomial (Looking at the limit of cones contains the element (a_1, \dots, a_n)).

References

- [1] P. L. Clark, *Geometry of Numbers with Applications to Number Theory*, Notes available at www.math.uga.edu/pete/geometryofnumbers.pdf.
- [2] M. Duchin, S. Lelivre, C. Mooney, *The geometry of spheres in free abelian groups*, Geometriae Dedicata 161.1, (2012).
- [3] E. Ehrhart, *Sur les polyedres rationnels homothetiques a n dimensions*, C.R. Acad. Sci. Paris, 254:616-618, (1962).
- [4] A. Ikeda, *Riemannian manifolds p-isospectral but not p + 1-isospectral*, Geometry of manifolds (Matsumoto, 1988), Perspect. Math. 8, 383-417, (1989).
- [5] A. Ikeda, *On the spectrum of a Riemannian manifold of positive constant curvature*, Osaka Journal of Mathematics, 17(1), 75-93, (1980).
- [6] E.A. Lauret, *Spectra of orbifolds with cyclic fundamental groups*, arXiv:1510.05948, (2015).

- [7] E.A. Lauret, R. J. Miatello, J. P. Rossetti, *Spectra of lens spaces from 1-norm spectra of congruence lattices*, International Mathematics Research Notices, (2015).
- [8] H. Mohades, B. Honari, *On a Relation between Spectral Theory of Lens Spaces and Ehrhart Theory*, arXiv:1601.04256, (2016).
- [9] R. P. Stanley, *Decompositions of rational convex polytope*, Ann. Discrete Math., 6:333-342, (1980).